Folded Solitary Waves and Foldons in the (2+1)-Dimensional Long Dispersive Wave Equation

J.-F. Zhang[a,b,c], Z.-M. Lu[a,b], and Y.-L. Liu[a]

[a] Shanghai Institute of Mathematics and Mechanics, Shanghai University, Shanghai 200072, P. R. China
[b] *Mailing address:* Department of Mathematical Sciences, Loughborough University, Loughborough, Leicestershire, LE11,3TU,UK
[c] Institute of Nonlinear Physics, Zhejiang Normal University, Jinhua 321004, P. R. China

Reprint requests to Dr, Z.-M. L.; E-mail: z.lu@lboro.ac.uk

Z. Naturforsch. **58a,** 280 – 284 (2003); received March 16, 2003

By means of the Bäcklund transformation, a quite general variable separation solution of the (2+1)-dimensional long dispersive wave equation: $\lambda q_t + q_{xx} - 2q\int(qr)_x \mathrm{d}y = 0$, $\lambda r_t - r_{xx} + 2r\int(qr)_x \mathrm{d}y = 0$, is derived. In addition to some types of the usual localized structures such as dromion, lumps, ring soliton and oscillated dromion, breathers soliton, fractal-dromion, peakon, compacton, fractal and chaotic soliton structures can be constructed by selecting the arbitrary single valued functions appropriately, a new class of localized coherent structures, that is the folded solitary waves and foldons, in this system are found by selecting appropriate multi-valuded functions. These structures exhibit interesting novel features not found in one-dimensions. – PACS: 03.40.Kf., 02.30.Jr, 03.65.Ge.

Key words: Variable Separation Solution; the (2+1)-dimensional Long Dispersive Wave System; Folded Solitary Wave; Foldon.

1. Introduction

In the study of (2+1)-dimensional nonlinear physical systems, much effort has been focused on single valued localized excitations, such as solitons, dromions, rings, lumps, breathers, instantons, peakons, compactons, localized chaotic and fractal patterns, etc. [1]. However, in nature, there exist very complicated folded phenomena such as the folded protein [2], the folded brain and skin surface, and many many other kinds of folded biological systems [3]. The bubbles on (or under) a fluid surface may be thought to be the simplest folded waves. Further, various kinds of ocean waves are folded waves also. Loop solitons, which are thought as a class of simplest folded waves in the (1+1)-dimensional case, have been founded in many (1+1)-dimensional integrable systems [4] and have been applied in physical fields like the string interaction with an external field [5], quantum field theory [6] and particle physics [7]. Recently, Tang and Lou [8] first considered these special folded localized excitations and found folded solitary waves in some (2+1)-dimensional nonlinear models, such as the (2+1)-dimensional dispersive long wave equation, the (2+1)-dimensional Burgers equation, etc. They define a new type of soliton, called the foldon if the interaction between the folded solitary waves is completely elastic. In this paper, we further consider the (2+1)-dimensional long dispersive wave equation (LDWE)

$$\lambda q_t + q_{xx} - 2q\int(pq)_x \mathrm{d}y = 0, \tag{1}$$

$$\lambda p_t - p_{xx} + 2p\int(pq)_x \mathrm{d}y = 0, \tag{2}$$

where $\partial_x = \partial_\xi$, $\partial_y = \partial_\xi - \lambda\partial_\eta$, and λ is a constant. This system apparently differs from the usual or traditional long dispersive wave system [9], which has been introduced by Chakravarty, Kent, and Newman [10] by symmetrical reduction from the self-dual Yang-Mills field equation. It is interesting to note that (1) and (2) can be reduced to a single nonlocal equation introduced by Fokas [11],

$$i\lambda q_t + q_{xx} - 2q\int |q|^2_x \mathrm{d}y = 0, \tag{3}$$

when $p = q^*$ and $t \to it$. Equation (3) arises in plasma physics under appropriate circumstances [12], and it admits exponentially localized solutions and satisfies the Painlevé property [13]. Radha and Lakshmanan investigated the Painlevé integrability properties and bilinear forms [14] of (1) and (2). Velan and Lakshmanan proved that (1) and (2) admit an infinite-dimensional symmetry and presented some physically interesting solutions via invariance analysis [15]. Starting from a special Bäcklund transformation, we converted the LDWE into simple variable separation equations and obtained a quite general variable separation solution [16]. Since some types of the usual localized excitations of (1) and (2), such as multi-soliton, multi-dromion, multi-lump, multi-ring soliton and oscillating dromion solutions, fractal dromion, fractal lump, and chaotic dromion have been obtained by selecting some types of lower-dimensional appropriate functions, here we try to find some kinds of folded solitary waves and foldons for (1) and (2).

2. Variable Separation Solution for the (2+1)-dimensional LDWE

For the concreteness and completeness of the investigation we review first the variables separation procedure for (1) and (2).

Making use of the transformation

$$pq = w_y, \tag{4}$$

where w is some arbitrary potential, (1) and (2) can be converted into the partial differential equations

$$\lambda q_t + q_{xx} - 2qw_x = 0, \tag{5}$$

$$\lambda p_t - p_{xx} + 2pw_x = 0, \tag{6}$$

$$pq = w_y, \tag{7}$$

where the integration constants in (5) and (6) have been set to be zero. By taking the Bäcklund transformation

$$q = \frac{Q}{f} + q_0, \quad p = \frac{P}{f} + p_0, \quad w = -(\ln f)_x + w_0, \tag{8}$$

which can be derived from the standard Painlevé truncated expansion, where f, P, Q are arbitrary differential functions of the arguments $\{x,y,t\}$, and (q_0, p_0, w_0) is an arbitrary seed solution, (5)–(7) become the following bilinear form

$$(D_x^2 + \lambda D_t)Q \cdot f + q_0 D_x^2 f \cdot f + f^2(q_{0xx} + \lambda q_{0t}) - 2fw_{0x}(Q + fq_0) = 0, \tag{9}$$

$$(D_x^2 - \lambda P_t)P \cdot f + p_0 D_x^2 f \cdot f + f^2(p_{0xx} - \lambda q_{0t}) - 2fw_{0x}(P + fp_0) = 0, \tag{10}$$

$$D_x D_y f \cdot f + 2(PQ + fPq_0 + fQp_0 + f^2 p_0 q_0 - f^2 w_{0y}) = 0, \tag{11}$$

where D_x, D_y, D_t are defined as

$$D_x^m D_y^n D_t^k f \cdot f = \lim_{x'=x, y'=y, t'=t} \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \cdot \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^n \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k f(x,y,t) \cdot f(x',y,'t'), \tag{12}$$

which are the usual bilinear operators introduced first by Hirota [17].

To discuss further, we take the seed solution (q_0, p_0, w_0) to be $q_0 = 0$, $p_0 = 0$, $w_0 = F_0(x,t)$. Then (9)–(11) can be simplified to

$$(D_x^2 + \lambda D_t)Q \cdot f - 2fQF_{0x} = 0, \tag{13}$$

$$(D_x^2 - \lambda D_t)P \cdot f - 2fPF_{0x} = 0, \tag{14}$$

$$D_x D_y f \cdot f + 2PQ = 0. \tag{15}$$

In order to find some interesting solutions of (13)–(15), we use the variable separation ansatz

$$f = a_1 F + a_2 G + a_3 FG, \; Q = F_1 G_1 \exp[\lambda(r+s)], \quad P = F_1 G_1 / \exp[\lambda(r+s)], \tag{16}$$

where a_1, a_2, a_3 are arbitrary constants and $F \equiv F(x,t)$, $G \equiv G(y,t)$, $F_1 \equiv F_1(x,t)$, $G_1 \equiv G_1(y,t)$, $r \equiv r(x,t)$, $s \equiv s(y,t)$ are all arbitrary functions of the indicated variables.

Substituting the ansatz (16) into (15) yields

$$F_1^2 G_1^2 - a_1 a_2 F_x G_y = 0. \tag{17}$$

Because the functions F and F_1 are only functions of $\{x,t\}$ and the functions G and G_1 are only functions of $\{y,t\}$, (17) can be solved by the following variable separated equations

$$F_1 = \delta_1 \sqrt{a_1 a_2 c_0^{-1} F_x}, \; G_1 = \delta_2 \sqrt{c_0 G_y}, \; (\delta_1^2 = \delta_2^2 = 1). \tag{18}$$

Similar to the above procedure, substituting (16) with (17) into (13) and (14) yields the following variable separated equations

$$-a_1 a_2 F_t = 2a_1 a_2 r_x F_x + c_1 F^2 + c_2 a_2 F + c_3 a_2^2, \tag{19}$$

$$a_1^2 G_t = (c_1 - c_2 a_3 + c_3 a_3^2) G^2 + a_1(-c_2 + 2c_3 a_3) G + a_1^2 c_3, \tag{20}$$

$$s = g(y) + b(t), \tag{21}$$

$$8F_x^2 F_{0x} = 2F_x F_{xxx} + 4\lambda^2 F_x^2 (r_x^2 + r_t + b_t) - F_{xx}^2, \tag{22}$$

where $g(y)$, $b(t)$ are arbitrary functions of the indicated variables.

Although it is still difficult to obtain general solutions of (19)–(22) for any fixed F_0, we can treat the problem in an alternative way. Because F_0 is an arbitrary seed solution, we can view F as an arbitrary function of $\{x,t\}$. The function r can be expressed by F simply by integration from (19). Then the seed solution F_0 can be fixed by (21). The result reads

$$r_x = \frac{-1}{2a_1 a_2 F_x}(c_1 F^2 + c_2 a_2 F + c_3 a_2^2 + a_1 a_2 F_t), \tag{23}$$

$$F_{0x} = \frac{1}{8F_x^2}[2F_x F_{xxx} + 4\lambda^2 F_x^2 (r_x^2 + r_t + b_t) - F_{xx}^2]. \tag{24}$$

As to the Riccati equation (20), its general solution has the form

$$G(y,t) = \frac{A_1(t)}{A_2(t) + U(y)} + A_3(t), \tag{25}$$

where $U(y)$ is an arbitrary function of y while A_1, A_2, and A_3 are arbitrary functions of t, which are linked with c_1, c_2, and c_3 by

$$c_1 = \frac{-1}{A_1}(2a_1 a_3 A_3 A_{2t} + a_1 a_3 A_{1t} + a_1^2 A_{2t} - a_3^2 A_1 A_{3t} + a_3^2 A_3^2 A_{2t} + a_3^2 A_3 A_{1t}), \tag{26}$$

$$c_2 = \frac{-1}{A_1}(2a_1 A_3 A_{2t} + a_1 A_{1t} - 2a_3 A_1 A_{3t} + 2a_3 A_3^2 A_{2t} + 2a_3 A_3 A_{1t}), \tag{27}$$

$$c_3 = \frac{-1}{A_1}(A_3 A_{1t} + A_3^2 A_{2t} - A_1 A_{3t}). \tag{28}$$

Using the relations (26)–(28), (20) becomes

$$G_t = \frac{-1}{A_1}\big[A_{2t} G^2 - (A_{1t} + 2A_3 A_{2t}) G + A_3^2 A_{2t} + A_3 A_{1t} - A_1 A_{3t}\big]. \tag{29}$$

One can verify directly that (25) is a general solution of (29).

Finally, substituting (16) with (18)–(22) into (9), we derive a quite general solution of the (2+1)-dimensional system (5)–(7):

$$q = \frac{\delta_1 \delta_2 \sqrt{a_1 a_2 F_x G_y}\exp[\lambda(r + g(y) + b(t))]}{a_1 F + a_2 G + a_3 F G}, \tag{30}$$

$$p = \frac{\delta_1 \delta_2 \sqrt{a_1 a_2 F_x G_y}}{(a_1 F + a_2 G + a_3 F G)\exp[\lambda(r + g(y) + b(t))]}, \tag{31}$$

$$w = F_0 - \frac{a_1 F_x + a_3 F_x G}{a_1 F + a_2 G + a_3 F G}, \tag{32}$$

with the four arbitrary functions $F(x,t)$, $G(y,t)$, $g(y)$, and $b(t)$, while F_0 and r are determined by (23) and (24). In the following discussion we study the structure of the potential qp. From the solutions (30) and (31) we have

$$qp = \frac{a_1 a_2 F_x G_y}{(a_1 F + a_2 G + a_3 F G)^2}. \tag{33}$$

3. Folded Solitary Waves and Foldons of the (2+1)-dimensional LDWE

It is of interest to mention that the solution (33) is valid for many (2+1)-dimensional models, and that many types of solutions can be obtained due to the arbitrariness of the functions F and G included in (33). Now we study the folded solitary waves and foldons. Starting from the the expression (33), these special solutions should be described by multi-valued functions. We write a localized function $\wp(x,t)$ in the form

$$\wp(x,t) = \sum_{j=1}^{M} f_j(\xi + v_j t), \; x = \xi + \sum_{i=1}^{M} g_j(\xi + v_j t), \tag{34}$$

where $v_1 < v_2 < \ldots v_M$ are all arbitrary constants and (f_j, g_j) and $\forall_j$ are all localized functions with $f_j(\pm\infty)$ and $g_i(\pm\infty) = G_i^{\pm}$ being constants. From the second equation of (34) we know that ξ may be a multi-valued function in some possible regions of x by selecting the function g_j suitably. So the function $\wp(x,t)$ may be a multi-valued function of x in these regions, though it is a single valued function of ξ. It is also clear that $\wp(x,t)$

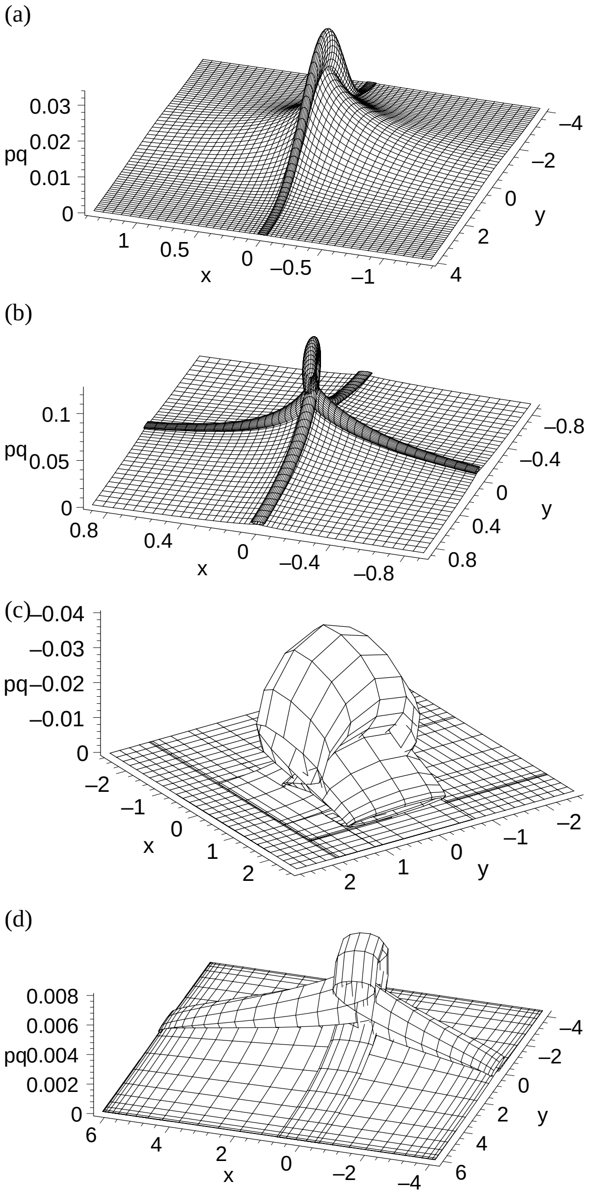

Fig. 1. Four typical folded solitary wave structures for the field pq expressed by (33) and $t=0$ in (34), (35). The related concrete choices are: (a) $F_x = \mathrm{sech}^2(\xi - v_1 t)$, $G_y = \mathrm{sech}^2\theta$, $x = \xi - 1.2\tanh(\xi - v_1 t), y = \theta + k_0 \tanh\theta, F = \int^{\xi} F_x x_{\xi}\mathrm{d}\xi + l_0, G = \int^{\theta} G_y y_{\theta}\mathrm{d}\theta + h_0$ and $a_1 = 1$, $a_2 = -2$, $a_3 = 0$, $k_0 = 1$, $l_0 = 8$, $h_0 = 0$; (b) same as (a) but with $k_0 = -1.1$; (c) F_x, F, G and a_1, a_2, a_3 are the same as (a), however $G_y = \mathrm{sech}^2\theta + \mathrm{sech}^6\theta$, $x = \xi + 2\tanh(\xi - v_1 t) + \tanh^2(\xi - v_1 t) - k_0\tanh^3(\xi - v_1 t)$, $y = \theta + 2\tanh\theta + \tanh^2(\theta) - k_0\tanh^3(\theta)$ and $l_0 = 50$, $h_0 = 50$, $k_0 = 5.5$; (d) same as (c) but with $h_0 = 250$, $k_0 = 10$.

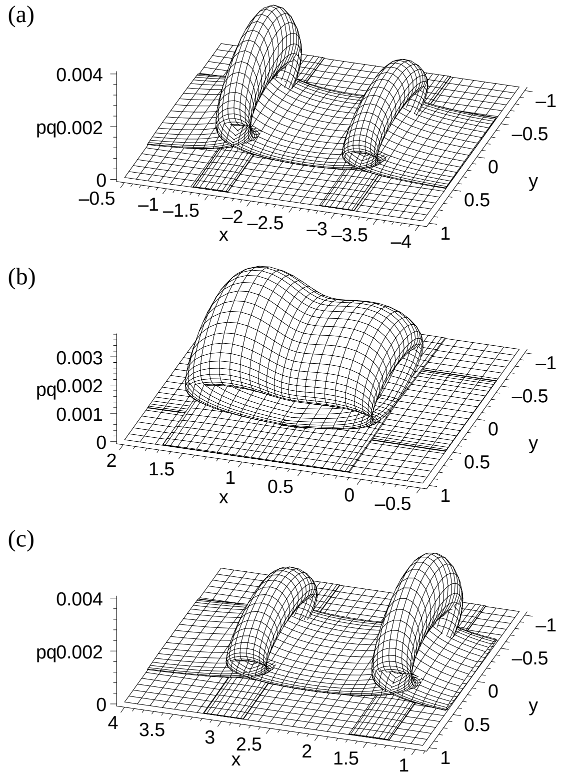

Fig. 2. Time evolution of two foldons for the field pq defined by (33) with $F_x = 0.8\mathrm{sech}^2\xi + 0.5\mathrm{sech}^2(\xi - 0.25t)$, $G_y = \mathrm{sech}^2\theta$, $x = \xi - 1.5\tanh\xi - 1.5 tanh(\xi - 0.25t)$, $y = \theta - 2\tanh\theta$, $F = \int^{\xi} F_x x_{\xi}\mathrm{d}\xi + 20$, $G = \int^{\theta} G_y y_{\theta}\mathrm{d}\theta$ and $a_1 = 1$, $a_2 = -2$, $a_3 = 0$. (a) $t = -18$, (b) $t = 7.2$, (c) $t = 18$.

is an interaction solution of M localized solutions because of the property $\xi\mid_{x\to\infty}\to\infty$. Now, if we specify different functions appearing in the formula (33), then we can get various types of folded solitary waves and / or foldons.

In Fig. 1, four typical special folded solitary waves are plotted for the field quantity pq shown by (3) with $F_x = \wp(x,t), F = \int^{\xi} F_x x_{\xi} d\xi + l_0$ and the functions $G(y,t)$ being given in a similar way:

$$G_y = \sum_{j=1}^{M} Q_j(\theta),\ y = \theta + R(\theta),\ G = \int^{\theta} G_y y_{\theta}\mathrm{d}\theta + h_0, \tag{35}$$

where l_0 and h_0 are arbitrary integration constants. In (35),$Q_j(\theta)$, $\forall j$, and $R(\theta)$ are localized functions of θ. The more detailed choices of functions in the figures are directly given in the figure legends.

Figures 2a – c show the possible existence of foldons which are given by (33), and the concrete choice of the functions is given in the figure legends. From Fig. 2a and 2c we can see that the interaction of two foldons is completely elastic. Because one of the velocities of foldons has been chosen as zero, it can also be seen that there are phase shifts for the two foldons. Especially, before the interaction, the static foldon (the large one) is located at $x = -1.5$, and after the interaction the large foldon is shifted to $x = 1.5$.

4. Summary and Discussion

In summary, with the help of the Bäcklund transformation, a rather general variable separation solution for the(2+1)-dimensional LDWE is found. Starting from the formula for the quantity pq expressed by (33) of the (2+1)-dimensional LDWE, folded solitary waves and foldons have been obtained by choosing arbitrary functions appropriately. We can see that the foldons may be folded quite arbitrarily and complicatedly, and that they possess quite rich structures and interaction behaviors. Explicit phase shifts for the localized structures defined by (33) have been given. In the natural world there exist many complicated folded and / or multi-valued phenomena. Seeking folded solitary waves and foldons and solutions of their interaction in other high dimensional nonlinear models seems useful and should be studied further.

Acknowledgement

One of the authors (Zhang Jiefang) is in debt to thank for the useful discussions with Professor Lou Senyue. We also thank Professor Roger Grimshaw and Doctor Rod Halburd in Loughborough University for their help. One of the referee's careful correction of the manuscript is also appreciated. This work is supported by the Pao Yu-Kong and Pao Zhao-long Scholarship for Chinese Students Studying Abroad, the Foundation of "151 Talent Engineering" of Zhejiang Province and the Natural Science Foundation of China (Grant No. 10272072).

[1] X. Y. Tang, S. Y. Lou, and Y. Zhang, Phys. Rev. **E66**, 046601 (2002); S. Y. Lou, C. L. Chen, and X. Y. Tang, J. Math. Phys. **43**, 4078 (2002).
[2] S.C. Trewick, T.F. Henshaw , R.P. Hansinger, T. Lindahl, and B. Sedgwick, Nature London **419**, 174 (2002); S.W. Lockless, and R. Ranganathan, Science **268**, 295 (1999); P.A. Lindagard and H. Bohr, Phys. Rev. Lett. **77**, 779 (1996).
[3] M. B. Goodman, G. G. Ernstrom, D. S. Chelur, R. O. Hagan, C. A. Yao, and M. Chalfie, Nature London **415**, 1039 (2002); B. L. Maclnnis, and R. B. Campenot, Science **295**, 1536 (2002).
[4] V. O. Vakhnenko, J. Phys. A: Math. Gen. **25**, 4181 (1992); V. O. Vakhnenko and E. J. Parks, Nonlinearity **11**, 1457 (1998); A. J. Morrison, E. J. Parks, and V. O. Vakhnenko, ibid **12**, 1427(1999).
[5] H. Kakuhata and K. J. Konno, Phys. Soc. Japan **68**, 757 (1999).
[6] S. Matsutani, Mod. Phys. Lett. A **10**, 717 (1985); J. Geom. Phys. **43**, 146 (2002).
[7] M. Schleif and R. Wunsch, Eur. J. Phys. A **1**, 171 (1998); M. Schleif, R. Wunsch, and T. Meissner, Int. J. Mod. Phys. E, **7**, 121 (1998).
[8] X. Y. Tang and S. Y. Lou, 2002 arXiv. nlin. SI/0210009. **4**, 1.
[9] M. Boiti, J. J. P. Leop, and F. Pempinelli, Inverse Probl. **3**, 371 (1987).
[10] S. Chakravarty, S. L. Kent, and E. I. Newman, J. Math. Phys. **36**, 763 (1995).
[11] A. S. Fokas, Inverse Probl. **10** L, 19 (1994).
[12] S. Novikov, S. V. Manakov, L. P. Pitaevskii, and V. E. Zakharov, Theory of Solitons: The Inverse Scatering Method, Consultants Bureau, New York 1984.
[13] R. Radha and M. Lakshmanan, Chaos, Solitons and Fractals **8**, 17 (1997).
[14] R. Radha and M. Lakshmanan, J. Math. Phys. **A38**, 292 (1997).
[15] M. S. Velan and M. Lakshmanan, Nonlinear Math. Phys. **4**, 251 (1997).
[16] J. F. Zhang, C. L. Zheng, J. P. Meng, and J. P. Fang, Chin. Phys. Lett. **20**, 331 (2003); C. L. Zheng, J. M. Zhu, J. F. Zhang, and L. Q. Chen, Commun. Theor. Phys. **39**, 261 (2003).
[17] R. Hirota, Phys. Rev. Lett. **27**, 1192 (1971).